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Number theory

Fall Semester 2013

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Quizz #2

Due Friday november 1rst in recitation.

Problems:

1. Let p be a prime. Prove that if $a^2 \equiv b^2 \ mod \ p$ then either $a \equiv b \ mod \ p$ or $a \equiv -b \ mod \ p$.

2. Compute $\phi(17)$ where ϕ is the Euler function.

3. Find the remainder of 3^{48} upon the division by 17.

4. John needs to pay exactly \$2.05 (no rest!). He has only quarters and dimes. In how many ways can he pay?

5. Find all positive integers n such that $[4]_n[5]_n = [-4]_n$ and $[3]_n + [6]_n \neq [1]_n$ where $[a]_n$ denotes the congruence class of the integer a mod n.

Solution:

1. The congruence $a^2 \equiv b^2 \ mod \ p$ means that $p|(a^2 - b^2)$. Note that $a^2 - b^2 = (a-b)(a+b)$. Thus p divides the product $(a-b)(a+b)$. Since p is a prime, p must divide one of the factors. If $p|(a-b)$ then $a \equiv b \ mod \ p$ and if $p|(a+b)$ then $a \equiv -b \ mod \ p$.

2. The number 17 is a prime so $\phi(17) = 17 - 1 = 16$.

3. Since $48 = 16 \times 3$ and $gcd(3, 17) = 1$, then by Fermat's theorem,

$$3^{48} \equiv (3^{16})^3 \equiv 1 \ mod \ 17$$

4. If John pays using x dimes and y quarters then $10x + 25y = 205$. thus the problem asks about the number of the solutions in non-negative integers x, y of the equation $10x + 25y = 205$. Dividing by 5, we get an equivalent equation $2x + 5y = 41$. Using Euclid's algorithm, we find that $(2, 5) = 1$ and $2 \times (-2) + 5 \times 1 = 1$. Multiplying by 41, we get $2 \times (-82) + 5 \times 41 = 41$. All the solutions to the equation $2x + 5y = 41$ are then given by $x = -82 + 5t \geq 0$ and $41 - 2t$, $t \in \mathbb{Z}$. We need to determine for how many values of t both x and y are non-negative integer. In other words, we look for integers t such that $-82 + 5t \geq 0$ and $41 - 2t \geq 0$. The first inequality is equivalent to $t \geq 16.4$ and the second is equivalent to $t \geq 20.5$. The integers t which satisfy these inequalities are $17, 18, 19, 20$. Thus there are 4 solutions in non-negative integers, i.e. John can pay in 4 ways.

5. Two congruence classes $[a]_n$ and $[b]_n$ coincide if and only if n divides $a-b$. Since $[4]_n[5]_n = [20]_n$, the equality $[4]_n[5]_n = [-4]_n$ holds if $20-(-4) = 24$ is divisible by n. Since $[3]_n + [6]_n \neq [9]_n$, the inequality $[3]_n + [6]_n \neq [1]_n$ holds if $9-1 = 8$ is not divisible by n. The positive divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. We select this that do not divide 8: $3, 6, 12$ and 24.